Constructing Scattering Amplitudes
Lecture 1: QCD and the Spinor-Helicity Formalism

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Abstract
Scattering amplitudes take surprisingly simple forms in theories such as quantum chromodynamics (QCD) and general relativity. This simplicity indicates deep symmetry. Recently, it has become possible to explain some of this symmetry. I will describe these insights and show how to derive amplitudes efficiently and elegantly. Key new ideas involve using complexified momentum, exploring singular behavior, and seeking clues in so-called twistor geometry. Complete amplitudes can be produced recursively. This streamlined approach is being applied in searches for new physics in high-energy particle colliders.

1 Introduction
The theme of this course is the study of scattering amplitudes through their simplicity and singularities.

When written as functions of well chosen variables, formulas for scattering amplitudes take simpler forms than one would naively expect from thinking of sums of Feynman diagrams. This simplicity is clearest in supersymmetric Yang-Mills theory, but extends to pure Yang-Mills, the Standard Model, and even (super)gravity. Simplicity motivates why and how we compute them: new structures demand new insights, and lead to new computational methods.

Explicit computations of amplitudes are important for experimental studies, notably in hadron colliders; and more formally, for exploring deeper structures in field theories such as Yang-Mills and supergravity.
Precision calculations at hadron colliders

Hadron colliders such as the Tevatron and especially LHC have very large QCD backgrounds. In order to observe signals of new physics, both background and signal must be computed to high precision (on the order of 1%). This typically means computing to next-to-leading order (NLO) in the strong coupling constant, and in some cases to next-to-next-to-leading order as well. One- and two-loop computations are thus of particular interest. These higher-order computations also have the effect of reducing renormalization scale dependence.

The QCD Factorization Theorem states that thanks to asymptotic freedom, an infrared-safe, collinear-safe observable can be expressed as a convolution of parton distribution functions with hard scattering kernels.

\[ \sum_{a,b} \int_0^1 dx_1 dx_2 f_{a/h_1}(x_1, \mu_f) H_{ab}(Q; Q^2/\mu_f^2, \mu_f/\mu, \alpha_s(\mu)) f_{b/h_2}(x_2, \mu_f) \]

The parton distribution function \( f_{a/h_1}(x_1, \mu_f) \) is the probability density of finding parton \( a \) of momentum fraction \( x_1 \) in proton \( h_1 \) at energy scale \( \mu_f \). Therefore it is meaningful to compute cross sections in perturbative QCD.

\( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory

This is a very special theory. It is conformal, with maximal supersymmetry in four dimensions. It is integrable in the planar limit. It is involved in various dualities: most famously, AdS/CFT; relevant to this course, a duality with twistor string theory; and internal dualities among Wilson loops, amplitudes, and correlation functions that are still being discovered today. Some practitioners like to refer to it as “the harmonic oscillator of the 21st century.” We will come back to this subject in Lecture 3, but for now I just want to touch upon a couple of motivations for pushing the boundaries of amplitude calculations.

For years, the “cusp anomalous dimension” or “soft anomalous dimension” \( f(\lambda) \) has been a computational target. It is the scaling of twist-2 operators in the limit of large spin,

\[ \Delta(Tr[\sigma^a Z D^\alpha Z]) - S = f(\lambda) \log S + \mathcal{O}(S^0). \]

Weak coupling calculations have been done from QCD pdf’s and gluon amplitudes, while a strong coupling expansion comes from AdS/CFT, integrability considerations and an all-loop Bethe ansatz. The number has been matched through 4 loops, giving checks on proposals for integrable structures.

Another inspiration has been the “BDS ansatz” (Bern, Dixon, Smirnov), a conjecture for iterative structure,

\[ \log \left( \frac{A_n}{A_{\text{tree}}} \right) = \text{Div}_n + \frac{f(\lambda)}{4} a_1(k_1, \ldots, k_n) + h(\lambda) + nk(\lambda) \]

The formula was conjectured based on the expected divergences of the amplitude from soft & collinear limits. By now we know that it fails for \( n > 5 \), and it has
been instructive to see why it could be satisfied for \( n \geq 5 \) (new symmetries!) and how it fails for \( n > 5 \), since the difference, called the “remainder function” has been predicted in the \( n \to \infty \) limit from strong-coupling considerations. Specifically, AdS/CFT predicts that the amplitude can be computed from the classical action of the string worldsheet whose boundary is a certain polygon. This correspondence extends to weak coupling as well.

**Gravity amplitudes**

Amplitudes in Einstein gravity or supergravity are simple as well, though we might not expect it from looking at the Lagrangian. Starting from a string theory analysis by Kawai, Lewellen and Tye (KLT) relating closed string amplitudes and open string amplitudes, one can take the field theory limit and relate graviton and gauge field amplitudes. These KLT relations are valid through at least two loops in \( \mathcal{N} = 8 \) supergravity.

One basic question in gravity theories is, could \( \mathcal{N} = 8 \) supergravity actually be finite? Like \( \mathcal{N} = 4 \) SYM, it is special by virtue of having maximal supersymmetry and no additional field content. Indeed, supersymmetry plays a role in suppressing expected divergences, but supersymmetry arguments alone can only eliminate divergences through a certain number of loops. Depending on the sophistication of the argument, the number starts at 3 and (so far) goes up to 9. No one has performed a 9-loop computation, but thanks to this motivation, 4-loop amplitudes have been computed and further computations are underway. Along the way, new identities have been discovered enhancing our understanding of amplitudes in general or in \( \mathcal{N} = 4 \) SYM. One reason to think it is possible that the theory is ultimately finite is that its relation to \( \mathcal{N} = 4 \) SYM is even stronger than we currently understand.

**2 QCD at tree level**

Now we begin studying amplitudes in detail. To look at concrete examples, we choose QCD as a theory amenable to new techniques. However, the spinor-helicity formalism can be generalized to include other field content. Massless fields are straightforward, while spinors for massive fields are somewhat less clean and are less widely used. The extension to supersymmetric Yang-Mills theory is easy and we will use it later.

Our field content is the gluon, transforming in the adjoint representation of the gauge group \( SU(N) \), and quarks and antiquarks of assorted flavors, transforming in the fundamental or antifundamental representation. The Feynman rules are given in Figure 1.

**Recommended reading:** Most of this introductory lecture follows [1] closely. I refer you to those lectures for additional explanations and a complete list of references. I am also using material from [2], especially in using Weyl spinors rather than Dirac spinors.
\[ -ig f^{ab} \eta_{\mu\nu}(q - p)^{\mu} \eta_{\nu}^{\alpha}(q - k)^{\alpha} + \eta_{\mu}^{\alpha}(k - p)^{\alpha} \]

\[ -ig^2 f^{abc} f^{def} (\eta_{\lambda\nu} \eta_{\mu}^{\alpha} - \eta_{\lambda\mu} \eta_{\nu}^{\alpha}) \]

\[ -ig^2 f^{abc} f^{def} (\eta_{\lambda\nu} \eta_{\mu}^{\alpha} - \eta_{\lambda\mu} \eta_{\nu}^{\alpha}) \]

\[ -ig^2 f^{abc} f^{def} (\eta_{\lambda\nu} \eta_{\mu}^{\alpha} - \eta_{\lambda\mu} \eta_{\nu}^{\alpha}) \]

\[ -ig \gamma_{\mu} T^a_{ij} \]

\[ -i \delta^{ab}_{ij} \]

\[ \frac{i \delta^{ij}_{\mu}}{\nu} \]

Figure 1: Feynman rules for QCD in Lorentz (Feynman) gauge with massless quarks, omitting ghosts.

2.1 Color ordering

The SU(N) color algebra is generated by the \( N \times N \) traceless hermitian matrices \( T^a \), with the color index \( a \) taking values from 1 to \( N^2 - 1 \). They are normalized by \( \text{Tr}(T^a T^b) = \delta^{ab} \). The structure constants \( f^{abc} \) are defined by \([T^a, T^b] = i \sqrt{\frac{N}{2}} f^{abc} T^c\), from which it follows that

\[ f^{abc} = -\frac{i}{\sqrt{2}} \left( \text{Tr}(T^a T^b T^c) - \text{Tr}(T^a T^c T^b) \right) \]  

(1)

Gluon propagators conserve color through the factor \( \delta^{ab} \). The traces in the \( f^{abc} \) at the vertices can be merged by the Fierz identity,

\[ \sum_a (T^a)_i^{\alpha} (T^a)_k^{\beta} = \delta_i^{\alpha} \delta_k^{\beta} - \frac{1}{N} \delta_i^{\alpha} \delta_k^{\beta} \]

(2)

It follows that at tree level, all color factors combine to form a single trace factor for each term. (One can check that the terms with \( 1/N \) all cancel among themselves; this is guaranteed by the fact that this term would be absent if the gauge group were \( U(N) \) rather than \( SU(N) \), but the auxiliary photon field does not couple to gluons anyway.)

The color decomposition for gluon amplitudes at tree level is:

\[ A_n^{\text{tree}}(\{a_i, p_i, c_i\}) = \]

(3)
\[ g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}}) A(p_{\sigma(1)}, \epsilon_{\sigma(1)}, \ldots, p_{\sigma(n)}, \epsilon_{\sigma(n)}). \]

The sum is over all permutations of the gluon labels, with a quotient by \( Z_n \) because the trace is cyclically invariant, so all these permutations can be combined in the same term. The function \( A(p_i, \epsilon_i) \) of kinematic arguments only is called the “color-ordered partial amplitude.” Once we have performed the color decomposition, we will refer to this function simply as the “amplitude” of the process. For tree amplitudes involving quarks, analogous formulas can be derived where instead of a trace, the string of matrices will be terminated by the specific \( T^a \)'s in the quark-quark-gluon vertices.

One-loop amplitudes of gluons have double-trace terms as well as single-trace terms. Their color decomposition is:

\[ A_{1\text{-loop}}^n \left( \{a_i, p_i, \epsilon_i\} \right) = g^n \left[ \sum_{\sigma \in S_n/Z_n} N \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}}) A_{n;1}(p_{\sigma(1)}, \epsilon_{\sigma(1)}, \ldots, p_{\sigma(n)}, \epsilon_{\sigma(n)}) \right. \]
\[ + \sum_{c=2}^{\lceil n/2 \rceil + 1} \sum_{\sigma \in S_n/S_{n,c}} \text{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(c-1)}}) \text{Tr}(T^{a_{\sigma(c)}} \cdots T^{a_{\sigma(n)}}) \times A_{n;c}(p_{\sigma(1)}, \epsilon_{\sigma(1)}, \ldots, p_{\sigma(n)}, \epsilon_{\sigma(n)}) \left. \right]. \]

Here, the partial amplitude \( A_{n;1} \) multiplying the single-trace term is called the leading-color partial amplitude, and the \( A_{n;c} \) are called subleading-color partial amplitudes. There is a relation among these partial amplitudes, so that in fact it suffices to compute the leading-color partial amplitudes. The \( A_{n;c} \) are then fixed by the identity

\[ A_{n;c}(1, 2, \ldots, c - 1; c, c + 1, \ldots, n) = (-1)^n \sum_{\sigma \in \text{COP}\{\alpha\}\{\beta\}} A_{n;1}(\sigma), \]

where \( \{\alpha\} \) is the reverse-ordered set \( \{c - 1, c - 2, \ldots, 2, 1\} \), \( \{\beta\} \) is the ordered set \( \{c, c + 1, \ldots, n\} \), and \( \text{COP} \) denotes the cyclically-ordered permutations of \( \{1, \ldots, n\} \) preserving the cyclic orderings of \( \{\alpha\} \) and \( \{\beta\} \).

Having carried out the color decomposition, we now turn our attention to the (leading-color) partial amplitudes. The upshot is that we only need to consider planar diagrams for a given cyclic ordering of gluons. There are suitably defined “color-ordered Feynman rules” generating these partial amplitudes, given in Figure 2.

### 2.2 Spinor-helicity formalism

The color-ordered amplitudes are functions of the (null) momentum 4-vectors \( p_i \) and the polarization vectors \( \epsilon_i \). In the spinor-helicity formalism, these numbers will be exchanged for spinors and helicity labels. The motivation is that there is
\[
\begin{align*}
\eta_{\mu p} &= \frac{i}{\sqrt{2}} \left( \eta_{\mu p} (p - q)_\mu + \eta_{\nu p} (q - k)_\nu + \eta_{\mu \nu} (k - p)_\rho \right) \\
\eta_{\mu \rho} &= i \eta_{\mu \rho} - \frac{i}{2} (\eta_{\mu \nu} \eta_{\rho \lambda} + \eta_{\mu \lambda} \eta_{\nu \rho}) \\
\gamma_{\mu \nu} &= \frac{i}{\sqrt{2}} \gamma_{\mu} - \frac{i}{\sqrt{2}} \gamma_{\nu} \\
\gamma_{\mu} &= - \frac{i}{\sqrt{2}} \gamma_{\mu} \\
\eta_{\mu \nu} &= \frac{i}{\sqrt{2}} \eta_{\mu \nu} \\
\end{align*}
\]

Figure 2: Color-ordered Feynman rules in QCD. Momenta are directed outwards from the cubic vertex.
too much redundancy among momenta and polarizations. Given a momentum vector $p_i$ for an external gluon, we know that the polarization vectors must be transverse and therefore satisfy $\epsilon_i \cdot p_i = 0$, and moreover that the shift $\epsilon_i \rightarrow \epsilon_i + wp_i$ for constant $w$ is a gauge transformation that must leave the amplitude invariant. However, there is no natural choice of $\epsilon_i$ given these constraints. By expressing the momentum in terms of spinors, we will improve this situation. This section follows the exposition in [2].

The Lorentz group is locally isomorphic to $SL(2) \times SL(2)$, whose finite-dimensional representations are labeled by $(p, q)$ taking integer or half-integer values. The positive-chirality spinor representation is $(1/2, 0)$, while the negative-chirality spinor representation is $(0, 1/2)$. We write $\lambda_a$ for a positive-chirality spinor and $\tilde{\lambda}_{\dot{a}}$ for a negative-chirality spinor, where the labels $a$ and $\dot{a}$ take values 1, 2.

The vector representation of $SO(3,1)$ is the $(1/2, 1/2)$ representation of $SL(2) \times SL(2)$, so a momentum vector should be viewed as an object with one each of the positive and negative chirality spinor labels. To see this map explicitly, let us work in the chiral (Weyl) basis of gamma matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (-1, \vec{\sigma})$. Given a Lorentz vector $p_\mu$, define a 2x2 matrix

$$p_{a\dot{a}} = \sigma^\mu_{a\dot{a}} p_\mu = p_0 + \vec{\sigma} \cdot \vec{p}. \quad (7)$$

One can see that $p_\mu^\mu = \det(p_{a\dot{a}})$. Thus a null momentum vector $p$ is mapped to a 2x2 matrix whose rank is strictly less than 2 and can therefore be written in the form

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}. \quad (8)$$

For real-valued momentum vectors in $+---$ signature, the spinors $\lambda_a$ and $\tilde{\lambda}_{\dot{a}}$ are actually complex-valued, but they are complex conjugates of each other, up to a sign. Later on, we will find it indispensable to work with complex-valued momentum vectors, for which $\lambda_a$ and $\tilde{\lambda}_{\dot{a}}$ become completely independent.

Notice that the equation (8) does not give unique values for the spinors. It is always possible to exchange a constant factor between them. In real Minkowski space, since the spinors should be complex conjugates of each other, this factor can only be a complex phase. The choice of a specific pair of spinors for a given momentum vector is equivalent to the choice of a wavefunction for a spin 1/2 particle of that same momentum. (This is the starting point in the exposition of spinor-helicity in [1].)

Explicit forms for the spinors associated to a null vector $(p_0, p_1, p_2, p_3)$ are

$$\lambda_a = \frac{e^{i\theta}}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_1 - ip_2 \\ p_0 - p_3 \end{pmatrix}, \quad \tilde{\lambda}_{\dot{a}} = \frac{e^{-i\theta}}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_1 + ip_2 \\ p_0 - p_3 \end{pmatrix}, \quad (9)$$

where $e^{i\theta}$ is the freely chosen phase factor. The spinor indices are raised and lowered with the epsilon tensors $\epsilon_{ab}, \epsilon_{\dot{a}\dot{b}}$ and their inverses $\epsilon^{ab}, \epsilon^{\dot{a}\dot{b}}$.
2.3 Spinor Products and Notation

Two spinors of the same chirality can be contracted with the epsilon tensors. We use different shapes of brackets for the two chiralities, and define antisymmetric spinor products as

$$\epsilon^{ab}\lambda_a\mu_b \equiv \langle \lambda\mu \rangle = -\langle \mu\lambda \rangle,$$

$$\epsilon^{\dot{a}\dot{b}}\tilde{\lambda}_{\dot{a}}\tilde{\mu}_{\dot{b}} \equiv [\tilde{\mu}\tilde{\lambda}] = -[\tilde{\lambda}\tilde{\mu}].$$

Just as we had $$p \cdot p = \text{det}(p\dot{a}\dot{a})$$, it is easy to see that $$p \cdot q = \frac{1}{2}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}p_{a\dot{a}}q_{b\dot{b}}$$. If both these vectors are null, we find

$$p \cdot q = \frac{1}{2} \langle \lambda\mu \rangle [\tilde{\mu}\tilde{\lambda}].$$

In the literature, it is very common to find 4-component Dirac spinors rather than the 2-component Weyl spinors. We summarize the correspondence and shorthand below. The subscripts on the Dirac spinors indicate helicity.

<table>
<thead>
<tr>
<th>Weyl shorthand</th>
<th>Weyl spinor</th>
<th>Dirac shorthand</th>
<th>Dirac spinor pos. energy</th>
<th>Dirac spinor neg. energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>\langle i</td>
<td>\lambda_a(p_i)</td>
<td>i^+</td>
<td>u_+(p_i)</td>
<td>v_+(p_i)</td>
</tr>
<tr>
<td>\langle i</td>
<td>\tilde{\lambda}<em>{\dot{a}}(p</em>{\dot{i}})</td>
<td>i^-</td>
<td>u_-(p_i)</td>
<td>v_-(p_{\dot{i}})</td>
</tr>
<tr>
<td>\langle i</td>
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</tr>
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<td>\langle i</td>
<td>\tilde{\lambda}<em>{\dot{a}}(p</em>{\dot{i}})</td>
<td>\langle i^+</td>
<td>u_+(p_{\dot{i}})</td>
<td>v_-(p_i)</td>
</tr>
</tbody>
</table>

It is also important to be aware that there are two commonly used conventions for the square-bracket spinor product (for negative chiralities), which differ by a sign. Here we do our best to follow the more traditional “QCD” conventions, where

$$\langle ij \rangle [ji] = 2p_i \cdot p_j = (p_i + p_j)^2 \equiv s_{ij}.$$  

The opposite sign is used in [2] and many “twistor-inspired” papers that followed.

More complicated contractions of spinor indices are expressed by expanded spinor products. Notice that

$$p_i = \langle i \rangle [i] + \langle i | i \rangle.$$  

When any Lorentz vector is sandwiched between spinors, it is redundant to write the slash, although it is common to do so. Therefore it should be understood that

$$\langle i | P[j] = \lambda_a(p_i)P_{b\dot{b}}\tilde{\lambda}_{\dot{a}}(p_j)\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}$$

$$= [j] P[i].$$
Chirality implies that a product such as $\langle i | P | j \rangle$ does not exist; in Dirac spinor terminology, $\langle i^- | P | j^+ \rangle = 0$. Here, $P$ is an arbitrary Lorentz vector. In amplitude calculations, we will typically find $P$ that are the sum of external momenta. Since the external legs are cyclically ordered, we define for convenience

$$P_{i,j} = p_i + p_{i+1} + \cdots + p_j$$

(17)

where indices are taken modulo $n$, the number of legs. Then, for example, our notation implies that

$$\langle k | P_{i,j} | \ell \rangle = \sum_{r=i}^{j} \langle ar \rangle [rb].$$

(18)

Before coming back to the subject of polarization vectors and amplitudes, let us introduce one useful spinor identity, the Schouten identity:

$$0 = \langle ij \rangle \langle k\ell \rangle + \langle ik \rangle \langle \ell j \rangle + \langle i\ell \rangle \langle jk \rangle. \quad (19)$$

The Schouten identity follows from the fact that the spinors live in a 2-dimensional space.

### 2.4 Polarization vectors

Now that we have expressed the gluon momenta in terms of spinors, we can write the polarization vectors in a natural form for each of the two helicity states. For a gluon of momentum $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$, the polarization vectors are

$$
\epsilon_{a\dot{a}}^- = -\sqrt{2} \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{|\lambda \mu|}, \quad \epsilon_{a\dot{a}}^+ = -\sqrt{2} \frac{\mu_a \tilde{\lambda}_{\dot{a}}}{\langle \mu \lambda \rangle},
$$

(20)

where $\mu$ and $\tilde{\mu}$ are arbitrary reference spinors, as long as the denominators do not vanish. Since they are directly proportional to the spinors, it is clear that the transverse condition $\epsilon \cdot p = 0$ is satisfied. It is also easy to see that the freedom to choose $\mu$ and $\tilde{\mu}$ is the freedom of gauge.

Thus the polarization vectors satisfy the following relations,

$$
\epsilon_{a\dot{a}}^+ \lambda_a \tilde{\lambda}_{\dot{a}} = 0, \quad (21)
\epsilon_{a\dot{a}}^+ \epsilon_{\dot{a}a}^+ = 0, \quad (22)
\epsilon_{a\dot{a}}^- \epsilon_{\dot{a}a}^- = 0, \quad (23)
\epsilon_{a\dot{a}}^+ \epsilon_{\dot{a}a}^- = -1. \quad (24)
$$

### 2.5 Helicity Amplitudes

With the expressions for polarization vectors, we can show that the simplest classes of helicity amplitudes of gluons vanish,

$$
A(1^+, 2^+, 3^+, \ldots, n^+) = 0, \\
A(1^-, 2^+, 3^+, \ldots, n^+) = 0.
$$
Because cyclic relabelings are not distinct, the second equation covers any configuration with \((n - 1)\) gluons of positive helicity and one of negative helicity.

To see that these amplitudes vanish, notice that all Lorentz indices must be contracted in the final expression. An \(n\)-point tree amplitude has at most \(n - 2\) momentum vectors in each term, from the cubic interaction vertices. That means that at least two of the polarization vectors must be contracted with each other. We can make all such contractions vanish by choosing the reference spinors wisely. If \(\mu_i = \mu_j\), then \(\epsilon_i^{-} \cdot \epsilon_j^{+} = 0\). So we choose all the \(\mu\) for positive helicities to be identical. The contraction \(\epsilon_1^{-} \cdot \epsilon_j^{+}\) vanishes if \(\tilde{\mu}_1 = \tilde{\lambda}_j\) or \(\mu_j = \lambda_1\). We can make this choice of \(\mu_j\) for all the positive helicities in the second amplitude. Similar arguments show that \(A(1^-q, 2^+, 3^+, \ldots, n^+) = 0\).

Thus, the nonvanishing amplitudes start when at least two gluons have helicity opposite to the rest. These are called Maximally Helicity Violating (MHV) amplitudes.

Naturally, we have parity-conjugate vanishing relations for amplitudes with only negative-helicity gluons or a single positive-helicity gluon. Parity conjugation is one of several useful identities satisfied by helicity amplitudes.

Reflection:
\[
A(1, 2, \ldots, n) = (-1)^n A(n, \ldots, 2, 1)
\]  
(25)

Parity conjugation:
\[
A(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) = (-1)^n (A(1^{-h_1}, 2^{-h_2}, \ldots, n^{-h_n}))_{\{1, 2, \ldots, n\}}
\]  
(26)

Photon decoupling identity:
\[
0 = A^{\text{tree}}(1, 2, 3, \ldots, n) + A^{\text{tree}}(2, 1, 3, \ldots, n) + A^{\text{tree}}(2, 3, 1, \ldots, n) + \cdots + A^{\text{tree}}(2, 3, \ldots, 1, n)
\]  
(27)

This identity is derived by decoupling the non-existent photon, which would be present if the gauge group were \(U(N)\) instead of \(SU(N)\). Then we could put the identity matrix into the trace in the color decomposition formula (4), whose form is unchanged. But there is no photon in QCD, so this color structure multiplies a vanishing expression.

Finally, the scaling of polarization vectors gives a scaling property of the full amplitude. For each particle labeled by \(i\),
\[
\left( \lambda_i^{a} \frac{\partial}{\partial \lambda_i^{a}} - \tilde{\lambda}_i^{a} \frac{\partial}{\partial \tilde{\lambda}_i^{a}} \right) A(\lambda_i, \tilde{\lambda}_i, h_i) = -2h_i A(\lambda_i, \tilde{\lambda}_i, h_i).
\]  
(28)

As we think about computing amplitudes, we start for small values of \(n\). For \(n = 4\) and \(n = 5\), every nonvanishing amplitude is MHV or conjugate-MHV. Moreover, we can use the photon decoupling identity to limit our computations to amplitudes where the negative helicities are cyclically adjacent, i.e. \(A(-, -, +, +)\) and \(A(-, -, +, +, +)\). Through judicious choices of reference spinors for the polarization vectors, we can drastically reduce the number of diagrams to compute and carry out the computation by hand without too much trouble. But as the number of legs increases, we will find recursive techniques immensely helpful.
2.6 Recursion for off-shell currents (Berends-Giele)

The Berends-Giele recursion for currents [3] generates gluon amplitudes by taking a single external leg off shell. Define \( J_\mu(1, 2, \ldots, n) \) as the sum of Feynman diagrams where gluons 1, \ldots, \( n \) are on shell but there is one additional off-shell gluon with the uncontracted vector index \( \mu \). The current can be constructed recursively by noticing that the gluon labeled by \( \mu \) must be attached to either a cubic or a quartic vertex, and in either case, the vertex is contracted with similar currents involving fewer legs. See Figure 3.

\[
J_\mu(1, 2, \ldots, n) = -\frac{i}{P_{1,n}^2} \left[ \sum_{i=1}^{n-1} V_{3}^{\mu \nu \rho}(P_{1,i}, P_{i+1,n}) J_\nu(1, 2, \ldots, i) J_\rho(i+1, \ldots, n) \\
+ \sum_{j=i+1}^{n-2} \sum_{i=1}^{n-2} V_{4}^{\mu \nu \rho \sigma}(P_{1,i}, P_{i+1,j}) J_\nu(1, 2, \ldots, i) J_\rho(i+1, \ldots, j) J_\sigma(j+1, \ldots, n) \right]
\]

The off-shell current satisfies the current conservation identity,

\[
P_{1,\ldots,n}^\mu \cdot J_\mu(1, 2, \ldots, n) = 0. \tag{29}
\]

To construct the \((n+1)\)-point gluon amplitude from the current \( J_\mu(1, 2, \ldots, n) \), first amputate the propagator by multiplying by \( iP_{1,n}^2 \). Then, contract with \( \epsilon_{n+1}^\mu \), the polarization vector of either helicity. Finally, take the limit \( p_{n+1}^2 = P_{1,n}^2 \to 0 \).

The algorithm is unsurpassed for its numerical power. Analytically, closed-form expressions are available for the simplest helicity configurations:

\[
J_\mu(1^+, 2^+, \ldots, n^+) = \frac{\langle q| \sigma^\mu P_{1,n} |q \rangle}{\sqrt{2} \langle q| 12 \rangle \cdots \langle n-1, n | nq \rangle}, \tag{30}
\]

\[
J_\mu(1^-, 2^+, \ldots, n^+) = \frac{\langle q| \sigma^\mu P_{2,n} |q \rangle}{\sqrt{2} \langle 12 \rangle \cdots \langle n-1, n | n1 \rangle} \sum_{m=3}^{n} \frac{\langle 1m | 1| P_{1,m} \rangle}{P_{1,m-1}^2 P_{1,m}^2} \tag{31}
\]
These formulas were constructed from an ansatz that could be checked to satisfy the recursion.

From these results, we can confirm that amplitudes with all positive or one negative helicity vanish. More importantly, we can prove the formula conjectured by Parke and Taylor for MHV amplitudes [4]. If the negative-helicity gluons are labeled by \( j \) and \( k \), then the amplitude is

\[
A(1^+, \ldots, j^-, \ldots, k^-, \ldots, n^+) = i \frac{(jk)^4}{(12)(23)\cdots(n1)}.
\]  

(32)

With Berends-Giele recursion, complete analytic results for gluon amplitudes were given up through \( n = 7 \), and also the complete next-to-MHV series \( A(1^-, 2^-, 3^-, 4^+, \ldots, n^+) \) where the negative-helicity gluons are cyclically adjacent.

**Exercise:** By parity conjugation of the Parke-Taylor formula for 4 gluons, we can see that

\[
i \frac{(12)^4}{(12)(23)(34)(41)} = i \frac{[34]^4}{[12][23][34][41]}.
\]

If we didn’t know that this expression represented an amplitude, the equality would not be obvious. Prove it using spinor identities, given momentum conservation \( p_1 + p_2 + p_3 + p_4 = 0 \).

**References**


