Time & Frequency Fundamentals & the Hydrogen Maser

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Introduction: Why Atomic Clocks?

✓ Used as frequency standards in Metrology
✓ SI second defined in terms of Cesium frequency
✓ Used as clocks to generate international time scales
✓ Used as reference oscillators in navigation and telecommunication systems
✓ Atomic clocks R&D is a branch of atomic physics
✓ Time & Frequency is a branch of Metrology (Tools to characterize atomic frequency standards)
Summary

1. From Celestial Time to Atomic Time

2. Time & Frequency fundamentals

3. The Hydrogen Maser
Part -1- From Celestial Time to Atomic Time

1. From Celestial Time to Atomic Time

2. International Time scales

3. International Time Comparisons via satellite

4. Relativity and time metrology
Part -2- Time & Frequency Fundamentals

1. Model of the oscillator

2. RF spectrum of oscillator with high phase noise

3. Characterization of oscillators: spectral domain

4. Characterization of oscillators: time domain
Part -3- The hydrogen Maser

1. Credit to the inventors

2. Selected topics from the physics of the HM

3. Applications of the hydrogen maser
Part - 1 -

From Celestial Time to Atomic Time
Part -1- Summary

1. From Celestial Time to Atomic Time
2. International Time scales
3. International Time Comparisons via satellite
4. Relativity and time metrology
Chapter - 1 -

From Celestial Time to Atomic Time
What is a time scale?

A time scale is a « time variable » used to specify the epoch of events in a given field of application.

A time scale is defined by specifying:
- a « scale unit » (unit of time)
- an « origin » (epoch of the origin of the scale)

To actually measure time, one needs a « clock », i.e. a frequency standard based on a stable and periodic physical process that can be observed and counted.
What is a time scale?
What is a time scale?

The clock must be calibrated versus the scale unit (frequency calibration) and origin (time calibration) specified for a given time scale.

Only after calibration can the clock be used to determine the epoch of observed events in terms of a specified time scale (or multiple time scales!)

By « time scale » we usually mean both the clock and its scale calibration(s).
From celestial time to atomic time

Until 1960 the Universal Time (UT) second was defined as a specified fraction (1/86400) of the LOD (Length Of Day)

LOD (Length of Day) is defined by the rotation of the Earth and is measured by observing the apparent movement of the stars

This definition of time was simple...
From celestial time to atomic time

...actually not so simple

As a matter of fact there are several definitions of UT

UT0 is the traditional GMT (Greenwich mean solar time)

UT1 is the modern form of UT corrected for polar motion and currently used in celestial navigation

UT2 is UT1 further corrected for the seasonal variations of the Earth rotation
From celestial time to atomic time

By 1952 progress in astronomy was such that UT was recognized to be too irregular to be used as the time scale in precise celestial dynamics observations.

Therefore ET (Ephemeris Time) was introduced.

- ET second is defined as a fraction of the tropical year.
- ET is a « dynamical » time scale, i.e. a time scale based on the theory of the dynamics of the celestial bodies.
- ET is realized via observation of the apparent motion of the Sun with respect to the Earth as described by Newcomb’s theory of the Sun (Newton’s mechanics).
From celestial time to atomic time

In 1960 the ET second was adopted as the SI definition.

The ET second is the fraction 1/31556925.9747 of the tropical year for 1900 January 1 at 12 h Ephemeris Time.

Since it was not possible to observe the Sun motion with sufficient accuracy, it was necessary to calibrate the motion of the moon...

... and to use the Moon motion as the clock to « realize » the Ephemeris Time scale.
From celestial time to atomic time

In 1955 the first cesium atomic clocks became operational

From 1956 the very first atomic time scales were established by several laboratories (Bureau International de l’Heure (BIH) in Paris and USNO Washington D.C.)

In 1958 Markowitz et al. published the value of the resonant frequency of the cesium 133 atom measured in terms of the ET second

The measurement took 5 years (1954-1958) !!!
From celestial time to atomic time

FREQUENCY OF CESIUM IN TERMS
OF EPHEMERIS TIME

W. Markowitz and R. Glenn Hall,
United States Naval Observatory, Washington, D. C.

and

L. Essen and J.V.L. Parry,
National Physical Laboratory, Teddington, England
(Received July 7, 1958)
From celestial time to atomic time

**FIG. 1.** Comparisons of Ephemeris Time and Atomic Time with Universal Time.
From celestial time to atomic time

\[
\nu_E = 9192631770 \pm 20 \text{ cycles per second (of E.T.) at 1957.0.}
\]
From celestial time to atomic time

Mr Essen (right), Mr Parry (left) and their cesium clock at NPL in 1955
From celestial time to atomic time

In 1967 the SI second was redefined as the duration of

9,192,631,770 periods

of the transition between the two hyperfine levels of the ground state of the cesium 133 atom

(based on the Essen & Parry 1958 meas. vs ET second)

this definition is still in use today and now realized with cesium cold atoms fountains with an uncertainty < $1 \times 10^{-15}$
Chapter- 2 -

International Time Scales
International time scales

✓ Every month the BIPM collects data from 350 atomic clocks in national laboratories (BIPM = Bureau International des Poids et Mesures)

✓ A paper clock (EAL = Echelle Atomique Libre) is generated

✓ The scale unit of EAL is not exactly the SI second because the participating clocks (commercial cesium or hydrogen masers) are not primary frequency standards

✓ As a matter of fact EAL is a kind of weighted average of the participating clocks (ALGOS algorithm)
International time scales

- EAL is steered to the SI second on the Geoid
  (relativity: proper time depends on gravity $1 \times 10^{-16}$ 1/m)

- TAI is the resulting paper clock (EAL + steering)

- Calibration of EAL against the SI second is performed by means of the primary cesium standards located in national laboratories

- TAI is actually a « real time paper clock » realization of TT (Terrestrial Time = SI second on the Geoid) based on an ensemble of physical clocks

- TAI is published every month in « Circular T »
Chapter - 3 -

International time comparisons via satellite
# International time comparisons

**Publication of UTC-UTC\(k\) in Circular T by BIPM**

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International time comparisons

✓ GPS Common View (SC, P3,)
✓ GPS geodetic
✓ TWSTFT
International time comparisons

GPS navigation: time is a free bonus

✓ need 4 satellites
✓ GPS satellites broadcast their position and GPS system time
✓ solve for 4 unknowns: position \( \vec{r} = (x, y, z) \) and time \( t \)
✓ GPS receiver gives position & local clock time as a bonus
✓ \( t \) is time of local clock versus GPS system time scale

\[
|\vec{r} - \vec{r}_1| = c(t - t_1) \\
|\vec{r} - \vec{r}_2| = c(t - t_2) \\
|\vec{r} - \vec{r}_3| = c(t - t_3) \\
|\vec{r} - \vec{r}_4| = c(t - t_4)
\]
International time comparisons

Principle of GPS Common-View

- CGGTTS format defined by BIPM
- 13 minute observations
- CGGTTS file yields $t_x - t_s$
- Common-View assumes that two stations look at the same satellite at the same time
- Propagation delays $\tau_{SA}$ and $\tau_{SB}$ are known from the nav solution
- GPS system time $t_s$ broadcasted by satellite is a common term

$$(t_a - t_s) - (t_b - t_s) = (t_a - t_b)$$
International time comparisons

- most NMI’s have GPS CV receivers (one way technique)
- some NMI’s have TWSTFT stations (two-way technique)
- accuracy is about 1 ns for GPS CV TWSTFT
- calibration of TWSTFT is more stable on the long-term
- BIPM uses the UTC(k) comparisons between NMI’s to generate UTC and TAI
International time comparisons

Calibration of GPS P3 receivers

✓ CV P3 technique uses geodetic GPS receivers
✓ geodetic receivers are able to measure the carrier phase of both P1 precise code observations (carrier L1 = 1575.42 MHz) and P2 precise code observations (carrier L2 = 1227.60 MHz)
✓ because of the dispersion of the ionosphere the L1 and L2 carriers do not have the same propagation delay
✓ P3 observations are a linear combination of P1 and P2 observations, independent of the ionospheric delay
✓ both the P1 and P2 internal delays in receiver must be calibrated
International time comparisons

principle of differential calibration (geodetic P3 receivers)
- DUT and REF receivers refer to same clock
- DUT and REF receivers in same location (zero baseline)
- REF receiver is already calibrated (absolute internal delays)
- parameters of DUT receiver adjusted to agree

\[
t_a = t_b = UTC(CH)
\]

\[
(t_a - t_s) - (t_b - t_s) = (t_a - t_b) = 0
\]
International time comparisons

calibration of GPS P3 receivers

✓ definition of external and internal delays
✓ in standard coax cable $5\text{ ns} = 1\text{ m}$ (70% speed of light)
International time comparisons

✓ CV P1 P2 DUT-REF before calibration
✓ INT DLY P1 and INT DLY P2 from REF used as initial val.

avrg P1 = -6.9 ns
avrg P2 = +1.7 ns
International time comparisons

CV P1 P2 DUT-REF after calibration

![Graph showing time comparisons](image)
International time comparisons

CV P3 CGGTTS calibration

- REF DLY is delay between local time scale and RCVR input
- CAB DLY is delay of antenna cable
- INT DLY P1 and INT DLY P2 are the internal delays
Accuracy of international comparisons

- TWSTFT international links can be calibrated to ± 1 ns
- GPS links can be calibrated to ± 2 ns
- Traditionally the long-term stability of TWSTFT transceivers used to be much better than the stability of GPS receivers
- Also GPS links used to be usable only on short-baselines (thousands of km) because «common view» of the same satellite by both stations was needed.
- Progress is made every year and performance of GNSS links and TWSTFT links are comparable
Accuracy of international comparisons

- Time uncertainty of ±1 ns over an observation time of 24 h is equivalent to a frequency uncertainty of

\[ \Delta y = \frac{1 \text{ns}}{1 \text{d}} = 1 \times 10^{-14} \]

- So it takes 10 d to compare frequency standards to
  \[ 1 \times 10^{-15} \]

- So it takes 100 d to compare frequency standards to
  \[ 1 \times 10^{-16} \]

- With the best primary standards comparison via satellite becomes impractical and fiber optic links are now used instead
Chapter -4-

Relativity and time metrology
Relativity effects in GPS

✓ 1 m = 3 ns at speed of light $c$

✓ clock prediction error $\langle \varepsilon \rangle_{RMS} = \sqrt{2} \tau \sigma_y(\tau)$

✓ sat atomic clocks must have frequency stability $1 \times 10^{-13}$ over 8 hour upload interval to accumulate < 3 ns error

✓ pseudoranges corrected for propagation effects (troposphere, ionosphere) corrections in the few ns range

✓ pseudoranges corrected for relativity effects (Sagnac, 2nd order Doppler, etc.) relativistic effects in the $1 \times 10^{-10}$ range

Relativity and time metrology

✓ GNSS navigation computations assume a non rotating Earth-Centered Inertial (ECI) reference frame in order to simplify the handling of relativistic effects and use the CGT (Coordinated Geocentric Time) i.e. coordinate time at the center of mass of the Earth.
✓ The TAI (International Atomic Time) and UTC (Universal Time Coordinated) time scales generated by the BIPM are based on TT (Terrestrial Time) which is based on the SI second as realised by a cesium primary standard located on the Geoid.
✓ The definitions of CGT and TT are conventions adopted by the IAU (International Astronomical Union) to manage relativistic effects (IAU general assemblies of 1976, 1979 and 1991)
✓ GPS time is based on the TT second like TAI and UTC
Relativity in cesium primary standards

✔ SI second defined as the cesium frequency on the Geoid (TT = Terrestrial Time)
✔ General Relativity makes the second depend on the gravitational potential
✔ in the uncertainty budget of a primary cesium standard a correction must be applied if the standard is not located on the Geoid
✔ the relative frequency changes by $1 \times 10^{-16}$ per meter
✔ current cesium fountain primary standards have uncertainties $\sim 3 \times 10^{-16}$
✔ which means that the location vs the Geoid must be known to $< 1$ m
Part -2 -

Model of the Oscillator
Part -2- Summary

1. Model of the oscillator
2. RF spectrum of oscillator with high phase noise
3. Characterization of oscillators: spectral domain
4. Characterization of oscillators: time domain
Chapter -1-

Model of the Oscillator
Model of the oscillator

Introduction

✓ Frequency standards (i.e. atomic clocks, quartz oscillators, GPS disciplined clocks), in practice are black boxes generating standard signals

✓ For frequency an RF output (sinusoidal wave) set at a standard frequency (5 MHz, 10 MHz or 100 MHz) is used

✓ For time a 1-PPS (1 Pulse Per Second) (square wave) that marks the beginning of each second is used.

✓ The standard model of the oscillator is a sinusoidal signal, at a nominal frequency, affected by various noise sources typical of different types of oscillators
Model of the oscillator

Phasor model

\[ \psi(t) = A\gamma(t) \exp(j2\pi\nu_0 t) \]

Complex envelope

\[ \gamma(t) = p(t) + jq(t) \]

\[ \gamma(t) = [1 + \epsilon(t)]\exp j\phi(t) \]
Model of the oscillator

Low noise oscillator

\[ |\varphi(t)| \ll 1 \]

\[ \exp j\varphi(t) = 1 + j\varphi(t) \]

\[ p(t) = 1 + \varepsilon(t) \]

\[ q(t) = \varphi(t) \]
Model of the oscillator

Spectrum of the Low Noise Oscillator

Autocorrelation function of complex envelope

\[ R_{\gamma\gamma} = 1 + R_{\epsilon\epsilon} + R_{\phi\phi} \]

Power spectral density of oscillator signal

\[ S_{ss}^+ = \frac{A^2}{2} \left[ \delta(f - \nu_0) + S_{\epsilon\epsilon}(f - \nu_0) + S_{\phi\phi}(f - \nu_0) \right] \]
Model of the oscillator

Multiplicative noise

✓ A multiplicative noise source actually modulates the carrier in phase (amplitude noise) or in quadrature (phase noise) affecting directly the envelope

✓ It cannot be distinguished from the original amplitude and phase noise

$$\psi(t) = A\gamma(t)exp(j2\pi v_0 t)$$

$$\gamma(t) = 1 + \varepsilon(t) + \varepsilon'(t) + j[\varphi(t) + \varphi'(t)]$$
Model of the oscillator

Additive noise

✓ An additive noise source has no phase relationship with the carrier (for example thermal noise in electronics)

✓ It just adds to the original signal.

\[ s(t) = \text{Real} \{ A \gamma(t) \exp(j2\pi v_0 t) \} + n(t) \]
Model of the oscillator

Rice’s representation theorem

\[ n(t) \text{ is a band limited white noise process of density } N_0 \text{ and bandwidth } 2B \]

\[ n_p(t) \text{ and } n_q(t) \text{ are low-pass white noise processes of density } N_0 \text{ and bandwidth } B \]

\[ n(t) = \text{Real} \left\{ \sqrt{2} \left[ n_p(t) + jn_q(t) \right] \exp(j2\pi v_0 t) \right\} \]
Model of the oscillator

Rice’s representation theorem

\[ n_q(t) \quad n_p(t) \]
Model of the oscillator

Rice’s representation theorem

✓ equivalent amplitude or phase noise due to additive noise is given by the carrier-to-noise ratio in a 1 Hz bandwidth

\[
n(t) = \text{Real} \left\{ A \left[ \frac{\sqrt{2}}{A} n_p(t) + j \frac{\sqrt{2}}{A} n_q(t) \right] \exp(j2\pi v_0 t) \right\}
\]

\[
S_{\varphi\varphi}^+(f) = S_{\varepsilon\varepsilon}^+(f) = \frac{N_0}{\frac{1}{2} A^2} = \frac{1}{C/N_0}
\]
Model of the oscillator

✓ the time error process $x(t)$ can be defined as the normalised phase difference accumulated between an oscillator used as a clock and a reference oscillator considered as the reference clock

$$x(t) = \frac{\varphi(t)}{2\pi v_0}$$

✓ the time error function is the phase difference (radian) between two oscillators expressed as a time difference in /s between two clocks
Model of the oscillator

✓ The normalised instantaneous frequency deviation $y(t)$ is the derivative of the time error function $x(t)$

$$y(t) = \frac{dx(t)}{dt} = \frac{1}{2\pi v_0} \times \frac{d\varphi(t)}{dt}$$

✓ $y(t)$ is the fractional frequency difference between two oscillators used as frequency standards
Chapter -2-

RF Spectrum of Oscillator with high phase noise
Spectrum of high phase noise oscillator

✓ Assume a pure band-limited phase noise modulation, i.e. no amplitude noise

✓ Autocorrelation function of the complex envelope is a classical result in telecom theory (Middleton 1960)

\[ \gamma(t) = A \exp[j \varphi(t)] \]

\[ R_{\gamma \gamma}(\tau) = A^2 E\{\exp[j(\varphi(t) - \varphi(t + \tau))]) \} \]

\[ R_{\gamma \gamma}(\tau) = A^2 \exp[R_{\varphi \varphi}(0) - R_{\varphi \varphi}(\tau)] \]
Spectrum of high phase noise oscillator

✓ The autocorrelation function of the complex envelope can be decomposed into a carrier \( \exp[-R_{\phi\phi}(0)] \), an in-phase part \( R_{pp}(\tau) \) and a quadrature part \( R_{qq}(\tau) \):

\[
R_{\gamma\gamma}(\tau) = \exp[-R_{\phi\phi}(0)] + R_{pp}(\tau) + R_{qq}(\tau)
\]

\[
R_{pp}(\tau) = \exp[-R_{\phi\phi}(0)] \times \cosh[R_{\phi\phi}(\tau)] - 1
\]

\[
R_{qq}(\tau) = \exp[-R_{\phi\phi}(0)] \times \sinh[R_{\phi\phi}(\tau)]
\]

\[
S^+_{ss} = \frac{A^2}{2} [\exp[-R_{\phi\phi}(0)] \times \delta(f - v_0) + S_{pp}(f - v_0) + S_{qq}(f - v_0)]
\]
Spectrum of high phase noise oscillator

If the mean square value of the phase noise process is very small then the quadrature component of the RF spectrum is a translated version of the phase noise PSD

This is the low-noise oscillator case discussed before

\[
R_{pp}(\tau) \cong 0
\]

\[
R_{qq}(\tau) \cong S_{\varphi \varphi}(f)
\]

\[
R_{\varphi \varphi}(0) = 0.01\text{rad}^2
\]
Spectrum of high phase noise oscillator

✓ When the mean square value of the phase noise process becomes very large, both the in-phase and the quadrature components of the RF spectrum become Gaussian.

\[ R_{\phi\phi}(0) = 0.1 \text{ rad}^2 \]

\[ R_{\phi\phi}(0) = 1 \text{ rad}^2 \]

\[ R_{\phi\phi}(0) = 2 \text{ rad}^2 \]

\[ R_{\phi\phi}(0) = 10 \text{ rad}^2 \]
Conclusion

✓ If the MS value of the phase noise is small then the RF spectrum is an image of the PSD of the phase noise process

✓ If the MS value of the phase noise is large then the RF spectrum is an image of the statistical distribution of the Gaussian phase noise process

✓ This is a classical result in telecom theory (Woodward’s theorem)
Chapter -3-

Characterization of the Oscillator: Spectral Domain
Spectral domain

✓ synchronous detection of phase noise (or amplitude noise)

\[ s(t) \times k \left[ \frac{V}{rad} \right] \times \varphi(t) \]

\[ \sin(2\pi v_0 t) \]

local oscillator in quadrature (or in phase)
Spectral domain

✓ The detected signal \( k \left[ \frac{V}{rad} \right] \times \varphi(t) \) is a low-pass electrical signal in \([V]\) that can be analysed using a spectrum analyser.

✓ Applying proper calibration the output of the spectrum analyser can be interpreted as the power spectral density of the phase noise process \( S_{\varphi\varphi}(f) \) in units \( \left[ \frac{rad^2}{Hz} \right] \).
Spectral domain

✓ electronics engineers like to use the notion of spectral purity to specify the phase noise of oscillators

✓ $\mathcal{L}(f)$ is a definition of the «Spectral Purity» i.e. the PSD of the RF signal normalised by the average carrier power

$$
\mathcal{L}(f) = \frac{S_{ss}^+(f + \nu_0)}{\frac{1}{2} A^2} = \delta(f) + S_{\epsilon\epsilon}(f) + S_{\phi\phi}(f)
$$

✓ forgetting about the carrier and the amplitude noise…

$$
\mathcal{L}(f) = \frac{1}{2} S_{\phi\phi}^+(f)
$$

in units $[\text{dBc/Hz}]$
Spectral domain

✓ Example of phase noise specification by a manufacturer
Spectral domain

Polynomial model of noise sources in oscillators

\[ S_{yy}(f) = \sum_{\alpha=-2}^{\alpha=2} h_{\alpha} f^\alpha \]

\[ S_{xx}(f) = \frac{S_{yy}(f)}{(2\pi f)^2} = \frac{1}{(2\pi)^2} \sum_{\beta=-4}^{\beta=0} h_{\alpha} f^\beta \]

with \( \beta = \alpha - 2 \)

remember that frequency \( y(t) \) is the derivative of time \( x(t) \)
Spectral domain

✓ Polynomial model

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<td>$h_{-2} f^{-2}$</td>
<td>$\frac{1}{(2\pi)^2} h_{-2} f^{-4}$</td>
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Spectral domain

✓ slope -3 (-30 dB per decade) indicates flicker FM noise
✓ zero slope indicates white PM noise (S/N 150 dB in 1 Hz)
Spectral domain

- Oscillators have non-stationary random noise processes
- A PSD divergent at the origin means infinite RMS value, undefined mean value, undefined standard deviation

\[ \langle [x(t)]^2 \rangle = R_{xx}(0) = \int_0^\infty S_{xx}^+(f) \, df \]
Chapter - 4 -

Characterization of the Oscillator: Time Domain
Time Domain

✓ historically «time domain» measurements were first defined as statistics made on the discrete average frequency samples taken from a frequency counter

\[
y_k(n\tau_0) = \frac{1}{n\tau_0} \int_{t_0}^{t_0+(k+1)n\tau_0} y(t) dt
\]

\[
y_k(n\tau_0) = \frac{1}{n\tau_0} [x(t_0 + (k + 1)n\tau_0) - x(t_0 + k n\tau_0)]
\]
Time Domain

✓ however, for the purposes of analysis, much simpler to consider continuous processes

✓ consider a few linear operators:

✓ moving average

\[ u(t, \tau) = \frac{1}{\tau} \int_{t-\tau}^{t} u(t) dt \]

✓ first increment

\[ \Delta(\tau)\{u(t)\} = u(t) - u(t - \tau) \]

✓ N th increment

\[ \Delta^{(n)}\{u(t)\} = \Delta^{(n-1)}\{u(t)\} - \Delta^{(n-1)}\{u(t)\} \]
Time Domain

✓ each operator is linear and causal (linear filter)
✓ below are the transfer functions:

✓ moving average

$$|H(j2\pi f)|^2 = \frac{\sin^2(\pi f \tau)}{(\pi f \tau)^2}$$

✓ first increment

$$|H(j2\pi f)|^2 = 4\sin^2(\pi f \tau)$$
Time Domain

✓ time domain measurements are defined as mean square values (variances) or RMS values (deviations)

\[ s_i(t) \xrightarrow{} |H(j2\pi f)|^2 \xrightarrow{} s_0(t) \]

\[ S_{s_0s_0}^+ (f) = |H(j2\pi f)|^2 S_{s_is_i}^+ (f) \]

\[ \langle (s_0(t))^2 \rangle = \int_0^\infty |H(j2\pi f)|^2 S_{s_is_i}^+ (f) df \]
Time Domain

✓ the true variance $I^2(\tau)$

✓ in the early ‘60 people working with atomic clocks tried to characterize the frequency stability by computing the mean square value of the frequency samples coming out from a frequency counter

✓ does not work because the frequency is non stationary

Time Domain

✓ the true variance $I^2(\tau)$

$$I^2(\tau) = \langle (y_k(\tau))^2 \rangle$$

$$I^2(\tau) = \left\langle (y(t, \tau))^2 \right\rangle$$

$$I^2(\tau) = \int_0^\infty \left[ \frac{\sin^2(\pi f \tau)}{(\pi f \tau)^2} \right] S_{yy}^+(f) df$$
Time Domain

✓ the Allan variance (two-sample) $\sigma_y^2(\tau)$

$$\sigma_y^2(\tau) = \frac{1}{2} \langle [y_k(\tau) - y_{k-1}(\tau)]^2 \rangle$$

$$\sigma_y^2(\tau) = \frac{1}{2} \langle [\Delta(\tau)\{y(t, \tau)\}]^2 \rangle$$

$$\sigma_y^2(\tau) = 2 \int_0^\infty \left[ \frac{\sin^4(\pi f \tau)}{(\pi f \tau)^2} \right] S_{yy}^+(f) df$$
## Time domain

- Polynomial model and Allan variance

<table>
<thead>
<tr>
<th></th>
<th>$S_{yy}(f)$</th>
<th>$\sigma_{y}^{2}(\tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>White noise PM</td>
<td>$h_{2}f^{2}$</td>
<td>$\frac{3B}{(2\pi)^{2}}h_{2}\tau^{-2}$</td>
</tr>
<tr>
<td>Flicker noise PM</td>
<td>$h_{1}f$</td>
<td>$\frac{1.038 + 3ln(2\pi B \tau)}{(2\pi)^{2}}h_{1}\tau^{-2}$</td>
</tr>
<tr>
<td>White noise FM</td>
<td>$h_{0}$</td>
<td>$\frac{1}{2}h_{0}\tau^{-1}$</td>
</tr>
<tr>
<td>Flicker noise FM</td>
<td>$h_{-1}f^{-1}$</td>
<td>$2ln(2)h_{-1}$</td>
</tr>
<tr>
<td>Random walk FM</td>
<td>$h_{-2}f^{-2}$</td>
<td>$\frac{2}{3}\pi^{2}h_{-2}\tau$</td>
</tr>
</tbody>
</table>
Time domain

- example of Allan deviation (square root of Allan variance)
- slope -1/2: white frequency noise
Time domain

- example of Allan deviation (square root of Allan variance)
- zero slope: flicker of frequency (example of OCXO)
Time domain

✓ Application of the Allan deviation in Metrology

✓ The Allan deviation on a given time interval yields the uncertainty on an average frequency measurement taken on that interval

✓ For example a commercial cesium frequency standard which has an Allan deviation of $1 \times 10^{-13}$ over a one day averaging interval can be calibrated with that uncertainty

✓ (assuming that the reference frequency standard is better)
Time domain

✓ computation of the Allan deviation from the time samples

✓ 3 samples of $x(t)$ are sufficient to generate one sample of the Allan deviation

\[
y(t, \tau) = \frac{1}{\tau} \Delta(\tau)\{x(t)\}
\]

\[
\Delta(\tau)\{y(t, \tau)\} = \frac{1}{\tau} \Delta^{(2)}\{x(t)\}
\]

\[
\sigma_y^2(\tau) = \frac{1}{2} \times \frac{1}{\tau^2} \langle [x(t) - 2x(t - \tau) + x(t - 2\tau)]^2 \rangle
\]
Example of Allan deviation from $x(t)$
Example of Allan deviation from $x(t)$
Example of Allan deviation from $x(t)$
Example of Allan deviation from $x(t)$
Example of Allan deviation from $x(t)$
Example of Allan deviation from $x(t)$

![Graph showing Allan deviation over time](image-url)
Example of Allan deviation from $x(t)$
**Time domain**

- prediction of a time process  
  (another way to look at the Allan deviation)

- predictor

  \[ \hat{x}(t + \tau) = x(t) + \tau \times y(t, \tau) \]

- error on the prediction

  \[ \epsilon(t + \tau) = x(t + \tau) - \hat{x}(t + \tau) \]

  \[ \epsilon(t + \tau) = \Delta^{(2)}(\tau)\{x(t + \tau)\} \]

  \[ \langle [\epsilon(t)]^2 \rangle = 2\tau^2 \sigma_y^2(\tau) \]
Example of time process prediction

![Graph showing normalized TA(CH)-TAI vs MJD (d)]
Example of time process prediction
Example of time process prediction
Example of time process prediction

![Graph showing time process prediction](image)
Example of time process prediction

![Graph showing normalised TA(Ch)-TAI over MJD/d]
Example of time process prediction
Part - 3 -

The Hydrogen Maser
Summary

1. Credit to the inventors

2. Selected topics from the physics of the HM

3. Applications of the hydrogen maser
The hydrogen Maser

passive hydrogen maser for Galileo (2014)

Varian hydrogen maser (1965)
The hydrogen Maser

✓ The first hydrogen maser was developed at Harvard in 1960

✓ First reported by Goldenberg, Kleppner and Ramsey in Physical Review Letters 5, 361 (1960)


✓ (Ramsey at Harvard 1952)
The hydrogen Maser

- Atomic frequency standard based on hydrogen

- MASER = Microwave Amplification by Stimulated Emission of Radiation

- Similar to a LASER but in the Microwave range

- Resonator is a microwave cavity resonant at 1420 MHz

- The hydrogen maser is an ACTIVE atomic clock. Stimulated emission of radiation constitutes an atomic microwave oscillator.

- Other atomic frequency standards are PASSIVE. The atomic resonance is used as a frequency discriminator.
The hydrogen Maser

- hyperfine splitting of hydrogen energy levels
- $mF = 0$ transition: 1’420’405’751 Hz
The hydrogen Maser

✓ Schematic hydrogen maser (source: Ramsey 1990)
The hydrogen Maser

✓ (Source: Ramsey 1965)
The hydrogen Maser

- cancel Doppler effect by storing hydrogen atoms into a box
- time constant determined by collimator
- avoid relaxation due to wall collisions (atomic hydrogen velocity 2 km/s) by coating the wall with Teflon

(source: US patent filed 1961)
The hydrogen Maser

✓ during the ‘60s and ‘70’s a lot of effort was made to make the hydrogen maser a **primary standard** and to define the second on the basis of the hydrogen atom

✓ the fundamental problem of the hydrogen maser, as a primary standard, is that collisions on the wall of the storage bulb introduce a frequency shift

✓ the wall shift is impossible to describe in terms of fundamental constants (Teflon coating)

✓ HM not a primary standard but a very stable frequency std
The hydrogen Maser

✓ threshold effect: stimulated emission power in the microwave cavity must be larger than cavity losses

✓ actual threshold $I_{th} \cong 1 \times 10^{12}$ atoms/s

✓ Source: Ramsey, Metrologia 1965

$$I_{th} \left( \frac{1}{2} h \nu \right) = \frac{1}{Q} \frac{H^2}{8\pi} V_c 2\pi \nu$$
The hydrogen Maser

✅ sources of atomic relaxation: \( \gamma = \gamma_0 + \gamma_w + \gamma_m + \gamma_{SE} \)
- wall collisions
- Spin Exchange (SE) (collisions between atoms)
- magnetic inhomogeneity
- escape rate from the storage bulb

✅ the rate of SE relaxation depends on the atomic density in the storage bulb

\[ \gamma_0 = \frac{1}{T_b} = \frac{\bar{v}A_e}{4KV_b} \]
\[ N = \frac{I_t}{\gamma_0} \]
\[ \gamma_{SE} = KN \]

✅ Source: Ramsey, Metrologia 1965
The hydrogen Maser

- the storage bulb time constant $T_b$ must be optimized:
  - if too large, density is too high and SE relaxation occurs
  - if too small, atoms escape before stimulated emission

(source: Kleppner, Physical Review 138, 1965)
The hydrogen Maser

- cavity pulling effect \[ \Delta y_a = \frac{Q_c}{Q_a} \Delta y_c \approx 1 \times 10^{-5} \]

- atomic quality factor \[ Q_a = \frac{v_0}{\left(\frac{\gamma}{2\pi}\right)} \approx 1 \times 10^9 \]

- cavity quality factor \[ Q_c = \frac{v_0}{\Delta v} \approx 1 \times 10^4 \]

- for comparison in a passive atomic frequency standard
  cavity pulling is much less critical:

\[ \Delta y_a = \left(\frac{Q_c}{Q_a}\right)^2 \Delta y_c \approx 1 \times 10^{-10} \]
The hydrogen maser

✓ applications of the hydrogen maser

✓ a stability of $\sigma_y(1000s) < 1 \times 10^{-15}$ is achievable
✓ this is among the best stability performances in atomic frequency standards

✓ VLBI radio-astronomy and geodesy
✓ deep space tracking
✓ space experiments (ACES)
✓ navigation systems (passive maser, Galileo)
✓ local frequency standard to evaluate and compare cesium fountains
Application of hydrogen maser to VLBI

✓ VLBI = Very Long Base Interferometry

✓ VLBI is done with a network of radio-telescopes on a continental, or even worldwide scale

✓ The interferometric (phase coherent) use of the network is equivalent to have a single radio-telescope antenna of continental size or even the same diameter as the Earth

✓ Real time synchronisation at 1 ps is not possible

✓ A hydrogen maser generates a local time scale for each radio-telescope and synchronisation is performed after-the fact by post-processing the data in a correlator

\[ 1 \text{ps} = 1000 \text{s} \times 1 \times 10^{-15} \]
VLBI applications

✓ Astronomy

✓ Geodesy

✓ Continental drift

✓ Earth rotation (length of day, pole migration)
VLBI basic principle
VLBI comparison of achievable resolutions

Diffraction-limited resolution of an optical instrument:
(\(\lambda\): wavelength, \(D\): diameter) \(\theta \sim \lambda / D\)

**human eye:**
\(\lambda / D \approx 10^{-4}\) radian \(\approx 20\) arc seconds (in practice \(\approx 200\) arcsec)

**Large optical telescope:**
\(\lambda / D \approx 10^{-7}\) radian \(\approx 0.025\) arc seconds (in practice \(\approx\) arc seconds)
Limit: atmospheric turbulence

**Large radiotelescope:**
\(\lambda / D \approx 10^{-3}\) radian \(\approx 200\) arc seconds (\(\approx\) practice)

**Worldwide radiotelescope network (\(D\) ist= 10 '000 km):**
\(\lambda / \text{Dist} \approx 10^{-9}\) radian \(\approx 0.0002\) arc seconds (\(\approx\) practice)

**VLBI:** VERY LARGE BASELINE INTERFEROMETRY
Continental drift rates measured by VLBI
Westford (USA) vs Wettzell (Germany)
Polar motion measured via VLBI

Die Polreihe beschreibt die wechselnde Lage des Erdrotationsspols und wird mit einer Auflösung von 5 Tagen aus den Satellitenbeobachtungen mitbestimmt. Die Polbewegung resultiert hauptsächlich aus der Überlagerung zweier Kreisbewegungen mit Perioden von 14 bzw. 12 Monaten und Amplituden von 6 bzw. 3 m. Dazu kommt eine Wanderung der Mittellage von ca. 10 cm/a. Die aktuelle Pollage verbindet das mittlere mit dem momentanen erdfesten Bezugs- system.

Genauigkeit: Pollage $\pm 0.002^\circ = \pm 6$ cm
Polar motion measured via VLBI
«superluminal» quasars observed VLBI
The End